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ADAPTED SETS OF MEASURES AND INVARIANT FUNCTIONALS ON $L^p(G)$

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ABSTRACT. Let G be a locally compact group. If G is compact, let $L_0^p(G)$ denote the functions in $L^p(G)$ having zero Haar integral. Let $M^1(G)$ denote the probability measures on G and let $\mathscr{P}^1(G) = M^1(G) \cap L^1(G)$. If $S \subseteq M^1(G)$, let $\Delta(L^p(G), S)$ denote the subspace of $L^p(G)$ generated by functions of the form $f - \mu * f$, $f \in L^p(G)$, $\mu \in S$. If G is compact, $\Delta(L^p(G), S) \subseteq L_0^p(G)$. When G is compact, conditions are given on S which ensure that for some finite subset F of S, $\Delta(L^p(G), F) = L_0^p(G)$ for all $1 < p < \infty$. The finite subset F will then have the property that every F-invariant linear functional on $L^p(G)$ is a multiple of Haar measure. Some results of a contrary nature are presented for noncompact groups. For example, if $1 \leq p \leq \infty$, conditions are given upon G, and upon subsets S of $M^1(G)$ whose elements satisfy certain growth conditions, which ensure that $L^p(G)$ has discontinuous, S-invariant linear functionals. The results are applied to show that for $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ has an infinite, independent family of discontinuous translation invariant functionals which are not $\mathscr{P}^1(\mathbb{R})$-invariant.

1. INTRODUCTION

On the circle group $\mathbf{T}$ let $L_0^2(\mathbf{T}) = \{f : f \in L^2(\mathbf{T}) \text{ and } \int_{\mathbf{T}} f = 0\}$. If $\phi \geq 0$ on $\mathbf{T}$, $\int_{\mathbf{T}} \phi = 1$ and $\hat{\phi}$ denotes the Fourier Transform of ϕ, then $|\hat{\phi}|$ is bounded away from one on $\mathbf{Z} \cap \{0\}^c$. Consequently, the linear operator defined on $\ell^2(Z \cap \{0\}^c)$ by $d \to d - \hat{\phi} d$ is bounded and invertible. It then follows from the Riesz-Fischer Theorem that $L_0^2(\mathbf{T}) = \{f - \phi * f : f \in L^2(\mathbf{T})\}$. It is an easy consequence of this that any ϕ-invariant linear functional on $L^2(\mathbf{T})$ is a multiple of Lebesgue measure and is therefore continuous.

One purpose of this paper is to extend these results in several ways. The circle group is replaced by a general compact group G, $L_0^2(\mathbf{T})$ is replaced by $L_0^p(G) = \{f : f \in L^p(G) \text{ and } \int_G f = 0\}$ and instead of considering convolution by a single probability measure in $L^1(\mathbf{T})$, as above, convolutions by elements of some given family of probability measures on G are considered simultaneously. Here, the main result is the following.

Theorem. *Let G be a compact group with dual $\hat{G}$ and let S be a subset of $M^1(G)$ such that* (i) *the elements of S are not simultaneously supported by a*

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proper closed subgroup of G*, and* (ii) *some average of elements of* S *has a Fourier transform whose norm is bounded away from one on the complement of some finite subset of* $\hat{G}$*. Then there exist* $\mu_1, \mu_2, \dots, \mu_n \in S$ *so that for all* $1 < p < \infty$,

$$L_0^p(G) = \left\{ \sum_{i=1}^{n} (f_i - \mu_i * f_i) : f_1, f_2, \dots, f_n \in L^p(G) \right\}.$$

Also, any linear functional on $L^p(G)$ *which is* μ_i*-invariant for each* $1 \leq i \leq n$*, is a multiple of Haar measure and is therefore continuous.*

This type of result is an analogue for convolutions of earlier results which have been obtained for translations in $L^p(G)$. Among these is the result of G. Meisters and W. Schmidt [6] according to which every translation invariant linear form on $L^2(G)$ is continuous, for any compact connected abelian group G. On the other hand, Meisters showed in [7] that if G is compact and disconnected, or if G is noncompact, $L^2(G)$ may have discontinuous translation invariant linear functionals. If G is a given compact abelian group whose component of the identity is C, L. Bagget and G. Meisters [8, p. 436] proved that if every translation invariant linear functional on $L^2(G)$ is continuous then G/C has a finitely generated dense subgroup, and the converse of this statement was proved by B. Johnson in [4]. More recently, J. Bourgain [1] has proved that for $1 < p < \infty$, every translation invariant linear form on $L^p(\mathbf{T})$ is continuous.

In [16], G. Woodward proved that for a noncompact, σ-compact group, $L^1(G)$ admits translation invariant, discontinuous linear functionals. If it is further assumed that G is amenable, he proved that for $1 < p \leq \infty$, $L^p(G)$ also admits translation invariant, discontinuous linear functionals. On the other hand, if G is not amenable, G. Willis [15] has proved that there are no nonzero translation invariant linear functionals on $L^p(G)$ for $1 < p \leq \infty$.

A second purpose of this paper is to obtain analogues of these results of Woodward where translations on the group are replaced by convolutions by measures in $M^1(G)$ which satisfy certain growth conditions. These results are then applied to show that for each $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ has an infinite, linearly independent family of translation invariant, discontinuous linear functionals which are not $\mathscr{P}^1(\mathbb{R})$-invariant. This is proved by showing that the subspace of $L^p(\mathbb{R})$ spanned by $\{f - \delta_g * f : f \in L^p(\mathbb{R}) \text{ and } g \in \mathbb{R}\}$ is both contained and has infinite codimension in the subspace of $L^p(\mathbb{R})$ spanned by $\{f - \phi * f : f \in L^p(\mathbb{R}) \text{ and } \phi \in \mathscr{P}^1(\mathbb{R})\}$. Some notation to be used in the paper now follows.

Let $M(G)$ denote the regular Borel probability measures on G, let $M^1(G)$ denote the probability measures in $M(G)$ and let $\mathscr{P}^1(G) = M^1(G) \cap L^1(G)$. If $\mu \in M(G)$, $\tilde{\mu}$ is defined in $M(G)$ by $\tilde{\mu}(A) = \overline{\mu(A^{-1})}$. Let $S \subseteq M^1(G)$ and let X be a Banach space of functions on G or let X be $L^p(G)$ for some $1 \leq p \leq \infty$. Then X is said to be *S-invariant* if for all $\mu \in S$ and $f \in X$, the convolution $\mu * f$ is defined and is in X. In this case let

$$\Delta(X, S) = \left\{ \sum_{i=1}^{n} (f_i - \mu_i * f_i) : n \in \mathbb{N},\ \mu_i \in S \text{ and } f_i \in X \text{ for } 1 \le i \le n \right\}.$$

If $S = \{\delta_g : g \in G\}$ and X is S-invariant, then X is said to be *translation invariant*, and $\Delta(X, S)$ is denoted by $\Delta(X, G)$. Let X' denote the algebraic dual of X, and let X be S-invariant. Then an element T of X' is said to be *S-invariant* if $T(\mu * f) = T(f)$ for all $\mu \in S$ and $f \in X$. Clearly, T is S-invariant if and only if $T = 0$ on $\Delta(X, S)$. If X is translation invariant, a functional $T \in X'$ is said to be *translation invariant* if it is $\{\delta_g\}$-invariant for all $g \in G$. The set of all S-invariant functionals in X' is denoted by $I(X, S)$, and the set of all translation invariant functionals in X' is denoted by $I(X, G)$. Of course, $I(X, G) = I(X, \{\delta_g : g \in G\})$.

A particular left invariant Haar measure on the locally compact group G will be denoted by λ. The bounded continuous complex-valued functions on G will be denoted by $C(G)$, the functions in $C(G)$ which vanish at infinity will be denoted by $C_0(G)$, and the left uniformly continuous functions in $C(G)$ will be denoted by $CU(G)$. Each of the spaces $C_0(G)$, $CU(G)$, $C(G)$ and $L^p(G)$ for $1 \le p \le \infty$ is $M^1(G)$-invariant. The identity element of G is denoted by e.

2. The general setting

Let S be a subset of $M^1(G)$ and let X be a space $L^p(G)$ for some $1 \le p \le \infty$, or let X be an S-invariant Banach space of complex valued functions on G.

Proposition 1. *If $\Delta(X, S)$ is not closed in X, there are discontinuous S-invariant functionals on X.*

Proof. With trivial changes, this may be proved along the lines of [7, Theorem 1] (see also [16, Theorem A]).

Lemma 1. *Let S, S_1 and S_2 be subsets of $M^1(G)$, and let X be invariant under S, S_1 and S_2. Then the following hold:*

(i) *$\Delta(X, S) = \bigcap\{\text{Kernel of } T : T \in I(X, S)\}$, and*
(ii) *$I(X, S_1) \subseteq I(X, S_2)$ if and only if $\Delta(X, S_1) \supseteq \Delta(X, S_2)$.*

Proof. It is obvious that if $T \in X'$, then $T \in I(X, S)$ if and only if $T = 0$ on $\Delta(X, S)$. Also, if $h \in X \cap \Delta(X, S)^c$, there is $T_0 \in X'$ so that $T_0(h) \ne 0$ but $T_0 = 0$ on $\Delta(X, S)$. Then $T_0 \in I(X, S)$ but h does not belong to the kernel of T_0. Conclusions (i) and (ii) follow from these observations.

Proposition 2. *Let S_1 and S_2 be subsets of $M^1(G)$ such that for some $\nu \in S_1$, $\nu * S_2 \subseteq S_1$. Then $I(X, S_1) \subseteq I(X, S_2)$ and $\Delta(X, S_2) \subseteq \Delta(X, S_1)$.*

Proof. Let $T \in I(X, S_1)$, $\mu \in S_2$ and $f \in X$. Then $T(\mu * f) = T(\nu * \mu * f) = T(f)$, as $\nu * \mu \in S_1$. Hence, $T \in I(X, S_2)$ so that $I(X, S_1) \subseteq I(X, S_2)$. The remainder follows from (ii) of Lemma 1.

Corollary. *Let* $1 \leq p \leq \infty$. *Then* $\Delta(L^p(G), G) \subseteq \Delta(L^p(G), \mathscr{P}^1(G))$ *and* $I(L^p(G), G) \supseteq I(L^p(G), \mathscr{P}^1(G))$.

Proof. If $a \in G$ and $\phi \in \mathscr{P}^1(G)$, then $\phi * \delta_a \in \mathscr{P}^1(G)$. Hence Proposition 1 applies.

Definitions. A nonempty subset S of $M^1(G)$ is said to be *adapted* if $\mu(A) = 1$ for all $\mu \in S$ implies that the group generated by A is dense in G. This is equivalent to requiring that there is no proper closed subgroup of G which supports all elements of S. A single measure $\mu \in S$ is said to be adapted if $\{\mu\}$ is adapted.

If G is noncompact, a family $(A_\alpha)_{\alpha \in I}$ of subsets of G is called *dispersed* if for each $n \in \mathbb{N}$, and each compact subset K of G, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $A_{\alpha_i}K \cap A_{\alpha_j}K = \varnothing$ for all $1 \leq i < j \leq n$.

Proposition 3. *Let G be σ-compact and let S be a nonempty subset of $M^1(G)$. Then the following hold:*

(i) *if G is compact, S is adapted and $1 < p < \infty$, then $\Delta(L^p(G), S)$ is dense in $L^p_0(G)$, and the only continuous S-invariant linear functionals on $L^p(G)$ are the multiples of Haar measure,*

(ii) *if G is not compact, S is adapted and $1 < p < \infty$, then $\Delta(L^p(G), S)$ is dense in $L^p(G)$, and the only continuous S-invariant linear functional on $L^p(G)$ is 0,*

(iii) *if S is right translation invariant, then $\Delta(L^1(G), S)$ is dense in $L^1_0(G)$ and the only continuous S-invariant linear functionals on $L^1(G)$ are the multiples of Haar measure, and*

(iv) *if G is noncompact and if the supports of the measures in S form a dispersed family in G, then $\Delta(C_0(G), S)$ is dense in $C_0(G)$.*

Proof. The first statements in each of (i), (ii) and (iii) are contained in Theorem 3 of [9]. The proofs of the remaining statements in (i) and (ii) are similar to the following one for (iii). Choose $f \in L^1(G)$ so that $\int_G f\,d\lambda \neq 0$. Then for $g \in L^1(G)$,

$$g = \left(\int_G g d\lambda\right)\left(\int_G f\,d\lambda\right)^{-1} f + \left[g - \left(\int_G g\,d\lambda\right)\left(\int_G f\,d\lambda\right)^{-1} f\right].$$

The function in the square brackets is in $L^1_0(G)$. Hence, if L is a continuous S-invariant linear functional on $L^1(G)$, we have $L = 0$ on $L^1_0(G)$ by the first part of (iii), so that $L(g) = (\int_G f\,d\lambda)^{-1} L(f) \int_G g\,d\lambda$, for all $g \in L^1(G)$.

To prove (iv), let $f \in C_0(G)$ and $n \in \mathbb{N}$. Let K be a compact subset of G so that $|f(x)| < n^{-1}$, for all $x \in G \cap K^c$. Then choose $\mu_1, \mu_2, \ldots, \mu_n \in S$ so that if $A_1, A_2, \ldots, A_n$ respectively denote their supports, then $A_iK \cap A_jK = \varnothing$, for $i \neq j$. Then if $x \notin A_iK$,

$$|(\mu_i * f)(x)| \leq \int_{A_i} |f(s^{-1}x)| d\mu_i(s) \leq n^{-1}.$$

Hence, $|(\sum_{i=1}^n \mu_i * f)(x)| \leq 1$, if $x \notin \bigcup_{i=1}^n A_iK$. On the other hand, if $x \in \bigcup_{i=1}^n A_iK$, $x \in A_jK$ for a unique j, so in this case $|(\sum_{i=1}^n \mu_i * f)(x)| \leq (n-1)n^{-1} + \|f\|_\infty$. Hence, $\|\sum_{i=1}^n \mu_i * f\|_\infty \leq 1 + \|f\|_\infty$. If $\nu \in C_0(G)^*$ is S-invariant, we now have

$$|\nu(f)| = \left|\nu\left(n^{-1}\left(\sum_{i=1}^n \mu_i * f\right)\right)\right| \leq n^{-1}\|\nu\|(1 + \|f\|_\infty),$$

true for all n, so $\nu(f) = 0$. Hence, if $\nu \in C_0(G)^*$ and $\nu = 0$ on $\Delta(C_0(G), S)$, then $\nu = 0$. It follows that $\Delta(C_0(G), S)$ is dense in $C_0(G)$.

Remark. The proof of (iv) is an adaptation of [16, Lemma 1(i)], which applies when $S = \{\delta_g : g \in G\}$ (see also [10, pp. 237–238]).

3. Results on compact groups

If G is a locally compact group, let $\hat{G}$ denote the dual object of G, and let ι denote the trivial representation in $\hat{G}$. If $\sigma \in \hat{G}$, we regard σ as a particular continuous unitary representation of G in $B(H_\sigma)$, the bounded linear operators on a Hilbert space H_σ whose dimension is denoted by d_σ and whose unit ball is denoted by U_σ. If $\mu \in M(G)$ and $\sigma \in \hat{G}$, $\hat{\mu}(\sigma) \in B(H_\sigma)$ is given by the equation

$$\langle \hat{\mu}(\sigma)u, v\rangle = \int_G \langle \sigma(t)u, v\rangle \, d\mu(t), \quad \text{for all } u, v \in H_\sigma.$$

If $S \subseteq M^1(G)$, let $H(S)$ be defined by

$$H(S) = \left\{ n^{-1}\left(\sum_{i=1}^n \mu_i\right) : n \in \mathbb{N} \text{ and } \mu_i \in S \text{ for all } 1 \leq i \leq n \right\}.$$

Then $S \subseteq H(S)$.

Lemma 2. *Let G be a compact group and let S be a subset of $M^1(G)$. Then the following hold:*

(i) *If S is adapted, for each $\sigma \in \hat{G}$ with $\sigma \neq \iota$, there is $\nu \in H(S)$ such that $I_\sigma - \hat{\nu}(\sigma)$ is invertible on H_σ, and*

(ii) *if $\{\tilde{\mu} * \mu : \mu \in S\}$ is adapted, for each $\sigma \in \hat{G}$ with $\sigma \neq \iota$, there is $\nu \in H(S)$ such that $\|\hat{\nu}(\sigma)\| < 1$.*

Proof. (i) Let S be adapted and let $\sigma \in \hat{G}$ with $\sigma \neq \iota$. Now let $u \in H_\sigma$, $\|u\| = 1$ and $\hat{\mu}(\sigma)u = u$ for all $\mu \in S$. Then

$$(3.1) \quad \int_G (1 - \langle(\sigma(x))(u), u\rangle)\, d\mu(x) = \int_G (\langle u, u\rangle - \langle(\sigma(x))(u), u\rangle)\, d\mu(x) = 0,$$

for all $\mu \in S$. As $\mu \geq 0$, and as $x \to \langle(\sigma(x))(u), u\rangle$ is continuous on G, we deduce that

$$1 = \langle u, u\rangle = \langle(\sigma(x))(u), u\rangle,$$

for all x belonging to the support of μ, for any $\mu \in S$. Thus, for any such x, $(\sigma(x))(u)$ is a multiple of u. Using (3.1), it now follows that $(\sigma(x))(u) = u$, for all x belonging to the support of any $\mu \in S$. As S is adapted, $(\sigma(x))(u) = u$ for all $x \in G$, which contradicts the fact that $\sigma \neq \iota$ and σ is irreducible.

It now follows that if $\sigma \in \hat{G} \cap \{\iota\}^c$,

$$\bigcap\{\text{Kernel of } I_\sigma - \hat{\mu}(\sigma): \mu \in S\} = \{0\}.$$

Each kernel of $I_\sigma - \hat{\mu}(\sigma)$ is closed. Also, U_σ is compact, as compactness of G implies H_σ is finite dimensional. Hence there are $\mu_1, \ldots, \mu_n \in S$ so that

$$\bigcap\{\text{Kernel of } I_\sigma - \hat{\mu}_i(\sigma): i = 1, 2, \ldots, n\} = \{0\}.$$

Let $\nu = n^{-1}(\sum_{i=1}^n \mu_i) \in H(S)$. Then if $(I_\sigma - \hat{\nu}(\sigma))(u) = 0$, for some $u \in H_\sigma$ with $\|u\| = 1$, we have

$$u = n^{-1}\left(\sum_{i=1}^n \hat{\mu}_i(\sigma)u\right).$$

As u is an extreme point of U_σ, and as $\hat{\mu}(\sigma)U_\sigma \subseteq U_\sigma$ for all $\mu \in M^1(G)$, it follows that $u = \hat{\mu}_i(\sigma)u$, for all $1 \leq i \leq n$. Hence,

$$u \in \bigcap\{\text{Kernel of } I_\sigma - \hat{\mu}_i(\sigma): 1 \leq i \leq n\} = \{0\},$$

so $u = 0$. Hence $I_\sigma - \hat{\nu}(\sigma)$ is injective on H_σ. As H_σ is finite dimensional, $I_\sigma - \hat{\nu}(\sigma)$ is invertible on H_σ. This proves (i).

(ii) Let $\{\tilde{\mu} * \mu: \mu \in S\}$ be adapted and let $\sigma \in \hat{G} \cap \{\iota\}^c$. Then as in the proof of (i) it follows that there are $\mu_1, \ldots, \mu_n \in S$ so that

$$\bigcap\{\text{Kernel of } ((\tilde{\mu}_i * \mu_i)^\wedge(\sigma) - I_\sigma): 1 \leq i \leq n\} = \{0\}.$$

Hence, if $u \in H_\sigma$ and $\|u\| = 1$, then for some $i \in \{1, 2, \ldots, n\}$, $\|\hat{\mu}_i(\sigma)u\| < 1$. As H_σ is finite dimensional, U_σ is norm compact and it follows that $\|n^{-1}(\sum_{i=1}^n \hat{\mu}_i(\sigma))\| < 1$. Hence, if we let $\nu = n^{-1}(\sum_{i=1}^n \mu_i)$, $\nu \in H(S)$ and $\|\hat{\nu}(\sigma)\| < 1$.

Lemma 3. *Let S be a subset of $M^1(G)$. Then if $\{\tilde{\mu} * \mu: \mu \in S\}$ is adapted, S is adapted. Also, if S is adapted, if $S \subset \mathscr{P}^1(G)$ and if $e \in \bigcap\{$support of $\mu: \mu \in S\}$, then $\{\tilde{\mu} * \mu: \mu \in S\}$ is adapted.*

Proof. Let H be a closed subgroup of G so that $\mu(H) = 1$ for all $\mu \in S$. Then $(\tilde{\mu} * \mu)(H) = 1$, for all $\mu \in S$. Hence if $\{\tilde{\mu} * \mu: \mu \in S\}$ is adapted, so too is S.

On the other hand, let H be a closed subgroup of G so that $(\tilde{\mu} * \mu)(H) = 1$, for all $\mu \in S$. Then for each $\mu \in S$, $\mu(sH) = 1$ for μ-almost all $s \in G$. If $S \subseteq \mathscr{P}^1(G)$, $s \to \mu(sH)$ is continuous on G for all $\mu \in S$. Hence in this case $\mu(sH) = 1$ for all s in the support of μ, for each $\mu \in S$. If $e \in \bigcap\{\text{support of } \mu : \mu \in s\}$, we then have $\mu(H) = 1$ for all $\mu \in S$. Thus, under these assumptions, if S is adapted, so too is $\{\tilde{\mu} * \mu : \mu \in S\}$.

Remark. It is not true in general that if S is adapted, so is $\{\tilde{\mu} * \mu : \mu \in S\}$. For example, let H be a proper closed subgroup of G and let $x \in G$ be such that xH is not contained in a proper closed subgroup of G. Let ν be the Haar measure on H. Then $\delta_x * \nu$ may belong to $\mathscr{P}^1(G)$ and in this case $\delta_x * \nu$ is adapted. However, $(\delta_x * \nu)^{\sim} * (\delta_x * \nu) = \nu$, which is not adapted.

The following result uses some concepts and terminology from the interpolation theory of linear operators. Definitions of undefined terms and a description of the interpolation method may be found in [5, Chapter IV].

Lemma 4. *Let B be a vector space which is the direct sum of two vector subspaces C and D. Let $\pi : B \to C$ be the associated projection. Let $|||\cdot|||_0$ and $|||\cdot|||_1$ be two norms on B which are consistent, and let π be bounded in each of these norms. If X is a vector subspace of B and $0 < \alpha < 1$, let $|||\cdot|||_{X,\alpha}$ denote the norm obtained on X by interpolating between the restrictions to X of $|||\cdot|||_0$ and $|||\cdot|||_1$. Let $|||\cdot|||_{B,C,\alpha}$ denote the norm on the quotient B/C obtained in the usual way from the norm $|||\cdot|||_{B,\alpha}$ on B. Then the spaces $(D, |||\cdot|||_{D,\alpha})$, $(B/C, |||\cdot|||_{B,C,\alpha})$ and $(D, |||\cdot|||_{B,\alpha})$ are mutually isomorphic as normed vector spaces.*

Proof. The result is surely known, although I have no explicit reference. It can be proved in a routine if tedious manner from the assumptions and the definition of the interpolation procedure that $(D, |||\cdot|||_{D,\alpha})$ and $(B/C, |||\cdot|||_{B,C,\alpha})$ are isomorphic. The isomorphism of this latter space with $(D, |||\cdot|||_{B,\alpha})$ is then easily checked.

Theorem 1. *Let G be a compact group and let S be an adapted subset of $M^1(G)$. Suppose that there exist $0 < \delta < 1$, a finite subset F of $\hat{G}$ and $\nu \in H(S)$ such that $\|\hat{\nu}(\sigma\| \le \delta$ for all $\sigma \in \hat{G} \cap F^c$. Then there exist $\mu_1, \mu_2, \dots, \mu_n \in S$ so that for each $1 < p < \infty$,*

$$L_0^p(G) = \left\{ \sum_{i=1}^{n} (f_i - \mu_i * f_i) : f_1, f_2, \dots, f_n \in L^p(G) \right\}. \tag{3.2}$$

Also, if ψ is any linear functional on $L^p(G)$, for some $1 < p < \infty$, and such that $\psi(\mu_i * f) = \psi(f)$ for all $1 \le i \le n$ and $f \in L^p(G)$, then ψ is continuous on $L^p(G)$ and is a multiple of Haar measure.

Proof. For each $\sigma \in \hat{G}$, let $u_{ij}^{(\sigma)}$, $1 \le i, j \le d_\sigma$, be a set of coordinate functions for σ. Let $\mathscr{F}_\sigma$ (respectively, $\mathscr{F}_{\sigma,k}$ for $1 \le k \le d_\sigma$) be the subspace of $L^2(G)$ spanned by $u_{ij}^{(\sigma)}$, $1 \le i, j \le d_\sigma$ (respectively, $u_{ik}^{(\sigma)}$ for $1 \le i \le d_\sigma$). If $\rho_1, \rho_2, \dots, \rho_{d_\sigma}$ is an orthonormal basis for H_σ such that $u_{ij}^{(\sigma)}(x) = \langle \sigma(x)\rho_j, \rho_i \rangle$ for all $x \in G$ and all $1 \le i, j \le d_\sigma$, let $T_{\sigma,k} : \mathscr{F}_{\sigma,k} \to H_{\overline{\sigma}}$ be defined for $1 \leq k \leq d_\sigma$ by

$$T_{\sigma,k}\left(\sum_{i=1}^{d_\sigma} c_i\sqrt{d_\sigma}u_{ik}^{(\sigma)}\right) = \sum_{i=1}^{d_\sigma} c_i\rho_i.$$

Then $T_{\sigma,k}$ is a linear isometry, which shows that the left regular representation of G on $\mathscr{F}_{\sigma,k}$ is equivalent to $\overline{\sigma}$ [3, p. 36]. If $\mu \in M(G)$, let T_μ denote the operator obtained by convolution by μ. Then for $\mu \in M(G)$, $T_{\sigma,k} \circ T_\mu = \hat{\mu}(\overline{\sigma}) \circ T_{\sigma,k}$ on $\mathscr{F}_{\sigma,k}$, so that $\hat{\mu}(\overline{\sigma})$ corresponds, under this equivalence, to convolution by μ on $\mathscr{F}_{\sigma,k}$. Hence, $\|T_\mu\|_{\mathscr{F}_{\sigma,k}} = \|\hat{\mu}(\overline{\sigma})\|$, for $\sigma \in \hat{G}$ and $1 \leq k \leq d_\sigma$.

If F is a finite subset of $\hat{G}$ and $1 \leq p \leq \infty$, let

$$L_F^p(G) = \left\{f : f \in L^p(G) \text{ and } \int_G f h\, d\lambda = 0 \text{ for all } h \in \bigcup_{\sigma\in\overline{F}} \mathscr{F}_\sigma\right\},$$

and let $X_F(G) = \sum_{\sigma\in\overline{F}} \mathscr{F}_\sigma$. Then $X_F(G)$ is a finite dimensional (hence closed) subspace of $L^p(G)$. If $\mu \in M(G)$, both $L_F^p(G)$ and $X_F(G)$ are invariant under T_μ. Now define $\pi : L^\infty(G) \to X_F(G)$ by

$$\pi(f) = \sum_{\sigma\in\overline{F}}\left(\sum_{i,j=1}^{d_\sigma}\left(\int_G f u_{ij}^{(\sigma)}\, d\lambda\right)\overline{u_{ij}^{\sigma}}\right).$$

Then π is a linear projection from $L^\infty(G)$ onto $X_F(G)$ and its kernel is $L_F^\infty(G)$.

Let $2 < p < \infty$ and let r be chosen so that $p < r < \infty$. Let $|||\cdot|||_0$ be the restriction to $L^\infty(G)$ of the L^2-norm and let $|||\cdot|||_1$ be the corresponding restriction of the L^r-norm. The projection π is continuous in each one of these norms, and the norms are consistent. Choose $0 < \alpha < 1$ so that the norm on $L^\infty(G)$ resulting by interpolating between $|||\cdot|||_0$ and $|||\cdot|||_1$ is the L^p-norm. Now let S, F, ν and δ be as given in the theorem. Then $\iota \in F$. It follows from the Peter-Weyl theorem and the fact that $\|T_\nu\|_{\mathscr{F}_{\sigma,k}} = \|\hat{\nu}(\overline{\sigma})\|$, that if $h|A$ denotes the restriction of a function h to a subset of its domain, then

$$\|T_\nu|L_F^\infty(G)\|_{|||\cdot|||_0} \leq \delta < 1.$$

Hence

$$\begin{aligned}\|T_\nu|L_F^\infty(G)\|_{|||\cdot|||_{L_F^\infty(G),\alpha}} &\leq \|T_\nu|L_F^\infty(G)\|_{|||\cdot|||_0}^{1-\alpha}\|T_\nu|L_F^\infty(G)\|_{|||\cdot|||_1}^{\alpha}\\ &\leq \delta^{1-\alpha} < 1.\end{aligned}$$

Thus, $(I - T_\nu)|L_F^\infty(G)$ is bounded and invertible on the completion of $(L_F^\infty(G),\ |||\cdot|||_{L_F^\infty(G),\alpha})$. By taking in Lemma 4 the spaces $L^\infty(G)$, $X_F(G)$ and $L_F^\infty(G)$ for B, C and D respectively, we can now deduce that $I - T_\nu$ is a bounded invertible operator on $(L_F^p(G), \|\cdot\|_p)$.

Let $f \in L_0^p(G)$. Then we can write $f = f_0 + \sum_{\sigma \in F} f_\sigma$, where $f_0 \in L_F^p(G)$ and $f_\sigma \in \mathscr{F}_{\overline{\sigma}}$ for all $\sigma \in F (f_\imath = 0)$. Let $g_0 = (I - T_\nu)^{-1} f_0 \in L_F^p(G)$. If $\sigma \in F \cap \{\imath\}^c$, since S is adapted, by Lemma 2 and the remarks above, it follows that there is $\nu_\sigma \in H(S)$ so that $I - T_{\nu_\sigma}$ is invertible on $\mathscr{F}_{\overline{\sigma}}$. Let $g_\sigma = (I - T_{\nu_\sigma})^{-1} f_\sigma \in \mathscr{F}_{\overline{\sigma}}$. We now have

$$\begin{aligned} f &= (I - T_\nu) g_0 + \sum_{\sigma \in F \cap \{\imath\}^c} (I - T_{\nu_\sigma}) g_\sigma \\ &= (g_0 - \nu * g_0) + \sum_{\sigma \in F \cap \{\imath\}^c} (g_\sigma - \nu_\sigma * g_\sigma). \end{aligned}$$

It should be noted that the choice of ν and the ν_σ is independent of p and $f \in L_0^p(G)$. This proves (3.2) for $2 < p < \infty$ and also contains the proof with $p = 2$. If $1 < p < 2$, an interpolation argument similar to the one above suffices to prove the statement. The remaining part of the theorem follows immediately.

Corollary 1. *Let G be a compact group and let S be an adapted subset of $M^1(G)$ such that $S \cap P^1(G) \neq \varnothing$. Then there are $\mu_1, \dots, \mu_n \in S$ so that for all $1 < p < \infty$,*

$$L_0^p(G) = \left\{ \sum_{i=1}^n (f_i - \mu_i * f_i) : f_i \in L^p(G) \text{ for } 1 \le i \le n \right\}.$$

Also, any linear functional on $L^p(G)$ which is μ_i-invariant for all i, $1 \le i \le n$, is a multiple of Haar measure.

Proof. Let $\phi \in S \cap \mathscr{P}^1(G)$. Then $\hat{\phi} \in c_0(\hat{G})$ [3, p. 81]. It is clear from this that Theorem 1 applies.

Corollary 2. *Let G be a compact group with a finite number of components, let $1 < p < \infty$ and let $\phi \in \mathscr{P}^1(G)$ be such that $\int_C \phi \, d\lambda > 0$ for each component C of G. Then $L_0^p(G) = \{f - \phi * f : f \in L^p(G)\}$ and every ϕ-invariant linear functional on $L^p(G)$ is a multiple of Haar measure.*

Proof. We have, for any Borel subset B of G,

$$\int_B \tilde{\phi} * \phi \, d\lambda = \int_G \left(\int_{t^{-1}B} \phi(s) \, d\lambda(s) \right) \phi(t^{-1}) \, d\lambda(t).$$

If $\int_A \tilde{\phi} * \phi \, d\lambda = 1$, since $\tilde{\phi} * \phi \in \mathscr{P}^1(G)$, it follows that $\int_{A \cap C} \tilde{\phi} * \phi \, d\lambda > 0$ for each component C of G. In particular, $\int_{A \cap C_0} \tilde{\phi} * \phi \, d\lambda > 0$, where C_0 is the component of the identity. It follows from Proposition 2 of [9] that the group generated by $A \cap C_0$ equals C_0. Hence, the group generated by A contains C_0 and intersects each coset of C_0, so it must equal G. Hence $\tilde{\phi} * \phi$ is adapted. Now Lemmas 2(ii) and 3 show that Theorem 1 applies with $S = \{\phi\}$.

Corollary 3. *Let G be a compact abelian group, and let $\mu \in M^1(G)$ be such that $|\hat{\mu}|$ is bounded away from 1 on $\hat{G} \cap \{\imath\}^c$. Then for each $1 < p < \infty$,*

$$L_0^p(G) = \{f - \mu * f : f \in L^p(G)\}.$$

Any linear functional on $L^p(G)$ which is μ-invariant is a multiple of the Haar measure on G.

Proof. Since $|\hat{\mu}|$ is bounded away from 1 on $\hat{G} \cap \{\imath\}^c$, μ is not supported by a proper closed subgroup of G. Hence μ is adapted and Theorem 1 applies.

Remarks. 1. Corollary 2 implies that if $\phi \in \mathscr{P}^1(G)$, where G is a compact connected group, then $L_0^p(G) = \{f - \phi * f : L^p(G)\}$ for all $1 < p < \infty$. This is a substantial strengthening of the Corollary to Theorem 4 in [9], which only applies to such groups when the regular representation of G upon $L_0^2(G)$ does not weakly contain the trivial representation.

2. Corollary 3 may also be proved by the method given for $L_0^2(\mathbf{T})$ in the introduction, and then using interpolation theory.

3. Let H be a countably infinite dense subgroup of a compact abelian group G. Then the linear space spanned by $\{f - \delta_h * f : h \in H \text{ and } f \in L^p(G)\}$ is not closed in $L_0^p(G)$ for $1 < p < \infty$ [12 and 13, Theorem 15]. However, $\{\delta_h : h \in H\}$ is adapted, so this shows that the condition in Theorem 1 that $\|\hat{\nu}(\sigma)\| \leq \delta < 1$ for $\sigma \in \hat{G} \cap F^c$ cannot be dropped. On the other hand, as the results mentioned in the introduction concerning group translations show ([6, Theorem 1], for example), this condition is not always essential for a conclusion along the lines of Theorem 1.

4. Results on noncompact groups

In this section, conditions are given which ensure that certain types of discontinuous invariant functionals exist on various function spaces on noncompact groups. If A, K are relatively compact subsets of G, let

$$Z(A, K) = \{x : x \in A \text{ and } xA^{-1} \supseteq K\}.$$

Lemma 5. *Let K, C be relatively compact subsets of G. Then there is a relatively compact open subset A of G such that $A \supseteq C$ and $Z(A, K) \supseteq C$.*

Proof. Let A be relatively compact, open and such that $A^{-1} \supseteq C^{-1}K \cup C^{-1}$. Then $A \supseteq C$ and if $x \in C$, $xA^{-1} \supseteq K$ so that $x \in Z(A, K)$. Hence $Z(A, K) \supseteq C$.

Lemma 6. *Let $\mu \in M^1(G)$ and let K be a compact subset of G such that $\mu(K) = 1$. Let A be a Borel subset of G. Then the function $\chi_A - \mu * \chi_A$ is zero outside of $(A \cup KA) \cap Z(A, K)^c$.*

Proof. We have $(\mu * \chi_A)(x) = \int_G \chi_A(s^{-1}x)\,d\mu(s) = \int_K \chi_{xA^{-1}}\,d\mu$. Hence $\mu * \chi_A = 1$ on $Z(A, K)$, so $\chi_A - \mu * \chi_A = 0$ on $Z(A, K)$. Also, if $xA^{-1} \cap K = \varnothing$, then $\mu * \chi_A(x) = 0$. Hence $\chi_A - \mu * \chi_A = 0$ outside of $KA \cup A$. Thus, $\chi_A - \mu * \chi_A$ is zero outside $(A \cup KA) \cap Z(A, K)^c$, as required.

Definitions. If A is a Borel subset of G and $\mu \in M(G)$, let $\mu_A \in M(G)$ be given by $\mu_A(B) = \mu(A \cap B)\mu(A)^{-1}$, if $\mu(A) \neq 0$, and by $\mu_A = 0$, if $\mu(A) = 0$. Then if $\mu \in M^1(G)$ and $\mu(A) > 0$, $\mu_A \in M^1(G)$.

When G is a σ-compact group, let $\mathscr{S}(G)$ be the set of all sequences $\tau = (K_n)_{n=1}^{\infty}$ of relatively compact subsets of G such that:

(i) each set K_n is open,
(ii) $K_n \subseteq K_{n+1}$ for all n,
(iii) $K_n = K_n^{-1}$ for all n,
(iv) $e \in K_n$ for all n, and
(v) $G = \bigcup_{n=1}^{\infty} K_n$.

Since G is σ-compact, $\mathscr{S}(G) \neq \varnothing$. If $\tau \in \mathscr{S}(G)$, let

$$M(\tau) = \left\{\mu : \mu \in M^1(G) \text{ and } \sum_{n=1}^{\infty}(1 - \mu(K_n)) < \infty\right\}.$$

$M(\tau)$ is a convex subset of $M^1(G)$ and $M_c^1(G) \subseteq M(\tau)$, where $M_c^1(G)$ denotes the measures in $M^1(G)$ of compact support. Hence, $\Delta(L^1(G), G) \subseteq \Delta(L^1(G), M(\tau))$.

Theorem 2. *Let G be σ-compact but not compact, and let $\tau \in \mathscr{S}(G)$. Then $\Delta(L^1(G), M(\tau))$ is not closed in $L_0^1(G)$, and there is a discontinuous linear functional on $L^1(G)$ which is $M(\tau)$-invariant.*

Proof. Let $\tau = (K_n) \in \mathscr{S}(G)$. Let $L_n = G \cap K_n^c$, for $n \in \mathbb{N}$. Then $\mu = \mu(K_n)\mu_{K_n} + \mu(L_n)\mu_{L_n}$, for $\mu \in M^1(G)$ and $n \in \mathbb{N}$. Let $f \in L^1(G)$ and let A be a Borel subset of G. Then, for $\mu \in M^1(G)$,

$$\begin{aligned}\left|\int_A (f - \mu * f)\, d\lambda\right| &\le \left|\int_A (f - \mu_{K_n} * f) d\lambda\right| + \left|\mu(L_n)\int_A (\mu_{K_n} * f - \mu_{L_n} * f)\, d\lambda\right| \\ &\le \left|\int_G f(\chi_A - \tilde{\mu}_{K_n} * \chi_A)\, d\lambda\right| + 2\mu(L_n)\|f\|_1 \qquad (4.1)\\ &\le \int_{T_n} |f|\, d\lambda + 2\mu(L_n)\|f\|_1,\end{aligned}$$

where T_n is any Borel subset of G such that $\chi_A - \tilde{\mu}_{K_n} * \chi_A$ is zero outside of T_n.

Let V be an open relatively compact neighborhood of e. Define inductively a sequence of open, relatively compact subsets of G as follows: $A_1 = V$ and when $A_1, \dots, A_r$ have been defined, let A_{r+1} be open, relatively compact and such that $A_{r+1} \supseteq A_r$, $A_{r+1} \cap A_r^c$ contains some left translate of V and $Z(A_{r+1}, K_{r+1}) \supseteq K_r A_r$ (here Lemma 5 has been used).

Let $r \in \mathbb{N}$ and $\mu \in M^1(G)$. Put $T_r = K_r A_r \cap Z(A_r, K_r)^c$. As $e \in K_r$ and $K_r = K_r^{-1}$, we deduce from Lemma 6 that $\chi_{A_r} - \tilde{\mu}_{K_r} * \chi_{A_r}$ is zero outside T_r. Also, if $r > s$, $Z(A_r, K_r) \supseteq K_{r-1}A_{r-1} \supseteq T_s$, so that $T_r \cap T_s = \varnothing$. Hence, for any $\mu \in M^1(G)$, the functions in the sequence $(\chi_{A_n} - \tilde{\mu}_{K_n} * \chi_{A_n})_{n=1}^{\infty}$ are concentrated on pairwise disjoint sets.

Let $f \in \Delta(L^1(G), M(\tau))$, and let $f_1, \ldots, f_q \in L^1(G)$, $\mu_1, \ldots, \mu_q \in M(\tau)$ be such that $f = \sum_{i=1}^q (f_i - \mu_i * f_i)$. Let $(\mu_i)_{K_n}$ be denote by μ_{i,K_n}. We now have, using (4.1),

$$\left|\int_{A_n} f\,d\lambda\right| \le \sum_{i=1}^q \left|\int_{A_n} (f_i - \mu_i * f_i)\,d\lambda\right|$$
$$\le \sum_{i=1}^q \int_{T_n} |f_i|\,d\lambda + 2\sum_{i=1}^q (1-\mu_i(K_n))\|f_i\|_1.$$

Since the sets in $(T_n)_{n=1}^\infty$ are pairwise disjoint and $\sum_{n=1}^\infty (1-\mu_i(K_n)) < \infty$ for each $1 \le i \le q$, we now have

$$\sum_{n=1}^\infty \left|\int_{A_n} f\,d\lambda\right| \le \left(\sum_{i=1}^q \|f_i\|_1\right)\left(1 + 2\max_{1\le i\le q}\left(\sum_{n=1}^\infty (1-\mu_i(K_n))\right)\right) \tag{4.2}$$
$$< \infty, \quad \text{for all } f \in \Delta(L^1(G), M(\tau)).$$

Since $A_{n+1} \cap A_n^c$ contains a left translate of V, for each $n \in \mathbb{N}$ choose $a_n \in G$ so that $a_n V \subseteq A_n \cap A_{n-1}^c$ (if $n = 1$ let $a_1 = e$ and $A_0 = \varnothing$). Then the method of [16, p. 212] may now be used to construct $\phi \in L_0^1(G)$ as follows: $\phi(x) = \lambda(V)^{-1}$ for $x \in A_1$, $\phi(x) = -2^{-n}\lambda(V)^{-1}$ for $x \in a_{2^n}V$ and $n \ge 1$, and $\phi(x) = 0$ elsewhere. Then as in [16], $\sum_{n=1}^\infty |\int_{A_n} \phi\,d\lambda| = \infty$, so by (4.2) $\phi \notin \Delta(L^1(G), M(\tau))$. However, since $\Delta(L^1(G), G) \subseteq \Delta(L^1(G), M(\tau))$, it follows from Proposition 3(iii) that $\phi \in \overline{\Delta(L^1(G), M(\tau))} = L_0^1(G)$. Hence $\Delta(L^1(G), M(\tau))$ is not closed in $L^1(G)$. The proof of the theorem is completed by applying Proposition 1.

Remark. Theorem 2 leaves open the question of whether $\Delta(L^1(G), M^1(G)) = L_0^1(G)$. Equivalently, are there discontinuous $M^1(G)$-invariant linear functions on $L^1(G)$?

Definition. Let $\tau = (K_n) \in \mathscr{S}(G)$ and let $\beta = (\beta_n)$ be a decreasing sequence of strictly positive numbers with limit 0. Then define $M(\tau, \beta)$ to be the set of all measures μ in $M^1(G)$ such that, for some $K \in \mathbb{R}$ (depending on μ), $1 - \mu(K_n) \le K\beta_n$, for all $n \in \mathbb{N}$. $M(\tau, \beta)$ is a convex subset of $M^1(G)$ which contains $M_c^1(G)$. The following is an analogue for $L^p(G)$ $(1 < p \le \infty)$ of Theorem 2 which applied for $L^1(G)$.

Theorem 3. *Let G be noncompact, σ-compact and amenable. Let $\tau = (K_n) \in \mathscr{S}(G)$ and let $\beta = (\beta_n)$ be a decreasing sequence of strictly positive numbers whose limit is 0. Then the following hold:*

(i) *If $1 < r < p < \infty$ and $\beta \in l^r$, $\Delta(L^p(G), M(\tau, \beta))$ is not closed in $L^p(G)$, and there is a discontinuous linear functional on $L^p(G)$ which is $M(\tau, \beta)$-invariant, and*

(ii) *if X denotes any one of the spaces $C_0(G)$, $CU(G)$, $C(G)$, $L^\infty(G)$, then $\Delta(X, M(\tau, \beta))$ is not closed in X and there is a discontinuous, $M(\tau, \beta)$-invariant functional on X.*

Proof. Let $1 < r < p \leq \infty$ and $p^{-1} + q^{-1} = 1$. Let $\delta > 0$ be chosen so that $r/p + \delta < 1$ (if $p = \infty$, $q = 1$ and $r/p = 0$). As G is amenable, there is a sequence (V_n) of open, relatively compact subsets of G so that $K_n \subseteq V_n \subseteq V_{n+1}$ and $\lambda(s^{-1}V_n \Delta V_n) < \beta_n^{(r/p+\delta)q}\lambda(V_n)$, for all $n \in \mathbb{N}$ and $s \in K_n$.

Now if $f \in L^p(G)$ and $\mu \in M^1(G)$, a calculation along the lines of [16, p. 209] shows that

$$\left|\int_{V_n}(f - \mu_{K_n} * f)d\lambda\right| \leq \|f\|_p \beta_n^{r/s+\delta}\lambda(V_n)^{1/q}.$$

Also, if $\mu \in M(\tau, \beta)$ and $M > 0$ is chosen so that $1 - \mu(K_n) \leq M\beta_n$ for all n, and if we use the fact that $\mu = \mu(K_n)\mu_{K_n} + (1 - \mu(K_n))\mu_{G\cap K_n^c}$ (as in the proof of Theorem 2), we find that

$$\left|\int_{V_n}(f - \mu * f)d\lambda\right| \leq \|f\|_p \beta_n^{r/p+\delta}\lambda(V_n)^{1/q}\{1 + 2M\beta_n^{1-(r/p+\delta)}\}.$$

It follows that if $\mu \in M(\tau, \beta)$, there is a constant M' so that

$$\left|\int_{V_n}(f - \mu * f)d\lambda\right| \leq M'\|f\|_p \beta_n^{r/p+\delta}\lambda(V_n)^{1/q}, \quad \text{for all } n \in \mathbb{N} \text{ and } f \in L^p(G).$$

An easy consequence of this is that if $h \in \Delta(L^p(G), M(\tau, \beta))$, there is $L > 0$ so that

$$\left|\int_{V_n} h d\lambda\right| \leq L\beta_n^{r/p+\delta}\lambda(V_n)^{1/q}, \quad \text{for all } n. \tag{4.3}$$

Now consider the case where $1 < r < p < \infty$ and $\beta \in l^r$. Define g on G by $g(x) = \beta_1^{r/p}\lambda(V_1)^{-1/p}$ for $x \in V_1$, and $g(x) = \beta_n^{r/p}\lambda(V_n)^{-1/p}$ for $x \in V_n \cap V_{n-1}^c$ and $n \geq 2$. Then by a similar argument to [16, p. 210], it follows that $g \in L^p(G)$, so that

$$\left|\int_{V_n} g d\lambda\right| \geq \beta_n^{r/p}\lambda(V_n)^{1/q}, \quad \text{for all } n.$$

Comparing this with the preceding inequality, we see that

$$g \notin \Delta(L^p(G), M(\tau, \beta)).$$

Since $\Delta(L^p(G), M(\tau, \beta))$ is dense in $L^p(G)$ by Proposition 3(ii), we see that $\Delta(L^p(G), M(\tau, \beta))$ is not closed, and the rest of (i) follows from Proposition 1.

In this case where $p = \infty$, choose a decreasing sequence (θ_n) of strictly positive numbers so that $\lim_{n\to\infty}\theta_n^{-1}\beta_n^\delta = 0$. Let U be a symmetric, open, relatively compact neighborhood of e. Define $\psi(x) = \theta_1$ for $x \in UV_1$, and $\psi(x) = \theta_n$ for $x \in UV_n \cap UV_{n-1}^c$ and $n \geq 2$. Let $g = \lambda(U)^{-1}\chi_U$ and define $\phi \in C_0(G)$ by putting $\phi = g * \psi$. Similarly to [16, p. 210], $|\int_{V_n} \phi d\lambda| \geq \theta_n \lambda(V_n)$, and by (4.3) it follows that $\phi \notin \Delta(L^\infty(G), M(\tau, \beta))$. Part (ii) of the Theorem now follows easily using Proposition 1 and Proposition 3(iv).

Remarks. The proofs of Theorems 2 and 3 are based upon the approach taken by G. Woodward in proving corresponding results for the spaces $\Delta(L^p(G), G)$ [16, Theorems 1 and 2]. In the case of Theorem 3, substantial modification was required to the proof of Theorem 1 in [16], so a relatively detailed proof has been presented. In the case of Theorem 3, lesser modification was required, so the proof of Theorem 3 has been somewhat truncated.

2. Theorem 3 leaves it open as to whether there are discontinuous $M^1(G)$-invariant linear functionals on $L^p(G)$, for $1 < p < \infty$.

Lemma 7. *Let g be a nonzero, nonnegative measurable function on $\mathbb{R}$ which is integrable on each compact subset of $\mathbb{R}$ and vanishes on $(-\infty, 0)$. Let $a > 0$ and let $\psi \in \mathscr{P}^1(\mathbb{R})$ be such that for some $b \in \mathbb{R}$, $\psi = 0$ on $(-\infty, b)$. Then the following hold:*

(i) *if $b \geq 0$, $\int_{-a}^a (\psi * g)(t)dt \leq \int_{-a}^a g(t)\,dt$, and*
(ii) *$\psi * g = 0$ on $(-\infty, b)$.*

Proof. Part (i) follows easily from the fact that

$$\int_{-a}^{a} (\psi * g)(t)dt = \int_b^\infty \psi(s) \left[\int_{-s-a}^{-s+a} g(t)dt \right] ds,$$

and (ii) follows from the definition of convolution.

Theorem 4. *Let X denote any one of the Banach spaces $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$, $C_0(\mathbb{R})$, $CU(\mathbb{R})$, $C(\mathbb{R})$ or $L^\infty(\mathbb{R})$. Then $\Delta(X, \mathbb{R})$ is a subspace of $\Delta(X, \mathscr{P}^1(\mathbb{R}))$ and it has infinite codimension in $\Delta(X, \mathscr{P}^1(\mathbb{R}))$. There exists in X' an infinite, linearly independent family of translation invariant linear functionals, such that each of these is discontinuous and is not $\mathscr{P}^1(\mathbb{R})$-invariant.*

Proof. The Corollary to Proposition 2 shows that $\Delta(X, \mathbb{R})$ is a subspace of $\Delta(X, \mathscr{P}^1(\mathbb{R}))$. If n, $s \in \mathbb{N}$ let $K_{n,s} = (-n^s, n^s)$ and let $\tau_s = (K_{n,s})_{n=1}^\infty \in \mathscr{S}(\mathbb{R})$. Also, let $V = (-1, 1)$.

Case 1. $1 < p \leq \infty$. Let $1 < r < p$, let $\delta > 0$ be chosen so that $r/p + \delta < 1$, and let $p^{-1} + q^{-1} = 1$ (if $p = \infty$, $0 < \delta < 1$, $r/p = 0$ and $q = 1$). If $n, s \in \mathbb{N}$ let $V_{n,s} = (-c_{n,s}, c_{n,s})$, where $c_{n,s} = n^{(r/p+\delta)q+s}$. Finally, let $\beta = (\beta_n)_{n=1}^\infty$ where $\beta_n = n^{-1}$ for all n.

It is easy to check that $K_{n,s} \subseteq V_{n,s}$ and that

$$\lambda((-t + V_{n,s})\Delta V_{n,s}) \leq \beta_n^{(r/p+\delta)q} \lambda(V_{n,s})$$

for all $n \in \mathbb{N}$ and $t \in K_{n,s}$. It follows now from (4.3) in the proof of Theorem 3 that if X is any one of the given spaces except $L^1(\mathbb{R})$, for each $h \in \Delta(X, M(\tau_s, \beta))$ there is $L > 0$ so that

$$\left|\int_{V_{n,s}} h\,d\lambda\right| \leq L n^{s/q}, \quad \text{for all } n \in \mathbb{N}. \tag{4.4}$$

Now for $n, s \in \mathbb{N}$ let $\phi_{n,s}$ and ϕ_s be given in $\mathscr{P}^1(\mathbb{R})$ by

$$\phi_{n,s} = \chi_{(c_{n,s}, c_{n,s}+1)} \quad \text{and} \quad \phi_s = (\zeta(1+\delta/2))^{-1} \sum_{i=1}^{\infty} i^{-(1+\delta/2)} \phi_{i,s}.$$

Let g be any function as in Lemma 7. Applying this lemma, and observing that $c_{i,s} \geq c_{n,s}$ if and only if $i \geq n$ now gives

$$\begin{aligned}\int_{V_{n,s}} (g - \phi_s * g)\,d\lambda &= (\zeta(1+\delta/2))^{-1} \sum_{i=1}^{\infty} i^{-(1+\delta/2)} \int_{-c_{n,s}}^{c_{n,s}} (g - \phi_{i,s} * g)(t)\,dt \\ &\geq (\zeta(1+\delta/2))^{-1} \left(\sum_{i=n}^{\infty} i^{-(1+\delta/2)}\right) \left(\int_0^{c_{n,s}} g(t)\,dt\right) \\ &\geq (\zeta(1+\delta/2))^{-1} 2\delta^{-1} n^{-\delta/2} \int_0^{c_{n,s}} g(t)\,dt.\end{aligned} \tag{4.5}$$

Case IA. $1 < p < \infty$ and $X = L^p(\mathbb{R})$. For $s \in \mathbb{N}$ define g_s on $\mathbb{R}$ by letting

$$\begin{aligned} &g_s(x) = 1, \quad \text{for } x \in [0, 1), \\ &g_s(x) = n^{-r/p} c_{n,s}^{-1/p}, \quad \text{for } x \in [c_{n-1,s}, c_{n,s}), \ n \geq 2, \quad \text{and} \\ &g_s(x) = 0, \quad \text{for } x \in (-\infty, 0). \end{aligned}$$

Then $g_s \in L^p(\mathbb{R})$ since

$$\begin{aligned}\int_{\mathbb{R}} |g_s|^p\,d\lambda &= \sum_{n=1}^{\infty} n^{-r} c_{n,s}^{-1}(c_{n,s} - c_{n-1,s}) \\ &\leq \sum_{n=1}^{\infty} n^{-r} < \infty, \quad \text{as } r > 1.\end{aligned}$$

Also,

$$\int_{V_{n,s}} g_s\,d\lambda \geq n^{-r/p} c_{n,s}^{-1/p} c_{n,s} = n^{\delta + s/q}.$$

It now follows from (4.5) that for all $n \in \mathbb{N}$

$$\int_{V_{n,s}} (g_s - \phi_s * g_s)\,d\lambda \geq (\zeta(1+\delta/2))^{-1} 2\delta^{-1} n^{\delta/2 + s/q}.$$

Comparing this with (4.4) shows that for each $s \in \mathbb{N}$,

$$g_s - \phi_s * g_s \notin \Delta(L^p(\mathbb{R}), M(\tau_s, \beta)).$$

On the other hand, if $s_1, s_2 \in \mathbb{N}$, a routine calculation shows that $\phi_{s_1} \in M(\tau_{s_2}, \beta)$ if and only if $s_2 \geq 2\delta^{-1}(s_1 + (r/p + \delta)q)$.

Now let $s_1 \in \mathbb{N}$ and define $s_n \in \mathbb{N}$ by letting $s_n > 2\delta^{-1}(s_{n-1} + (r/p + \delta)q)$ for $n \geq 2$. Then the subspaces $\Delta(L^p(\mathbb{R}), M(\tau_{s_n}, \beta))$ of $L^p(\mathbb{R})$ increase as n increases. Also, for all $n \in \mathbb{N}$,

$$g_{s_n} - \phi_{s_n} * g_{s_n} \in \Delta(L^p(\mathbb{R}), M(\tau_{s_{n+1}}, \beta)) \cap (\Delta(L^p(\mathbb{R}), M(\tau_{s_n}, \beta)))^c.$$

It follows easily from these facts that no nontrivial linear combination of functions in $\{g_{s_n} - \phi_{s_n} * g_{s_n} : n \in \mathbb{N}\}$ belongs to $\Delta(L^p(\mathbb{R}), M(\tau_{s_1}, \beta))$ or to $\Delta(L^p(\mathbb{R}), \mathbb{R})$, since this latter subspace is smaller. Since each function $g_{s_n} - \phi_{s_n} * g_{s_n} \in \Delta(L^p(\mathbb{R}), \mathscr{P}^1(\mathbb{R}))$, it follows that $\Delta(L^p(\mathbb{R}), \mathbb{R})$ has infinite codimension in $\Delta(L^p(\mathbb{R}), \mathscr{P}^1(\mathbb{R}))$. For each $n \in \mathbb{N}$, there is $L_n \in (L^p(\mathbb{R}))'$ so that $L_n = 0$ on $\Delta(L^p(\mathbb{R}), \mathbb{R})$, $L_n(g_{s_n} - \phi_{s_n} * g_{s_n}) = 1$ and $L_n(g_{s_m} - \phi_{s_m} * g_{s_m}) = 0$ if $m \neq n$. Then $\{L_n : n \in \mathbb{N}\}$ is an independent family of translation invariant functionals which are not $\mathscr{P}^1(G)$-invariant. Proposition 3(ii) implies that each L_n is discontinuous. This proves the theorem when $X = L^p(\mathbb{R})$ and $1 < p < \infty$.

Case IB. $p = \infty$ and $X = C_0(\mathbb{R})$, $CU(\mathbb{R})$, $C(\mathbb{R})$ or $L^\infty(\mathbb{R})$. In this case let $h = 2^{-1}\chi_V = 2^{-1}\chi_{(-1,1)} \in \mathscr{P}^1(\mathbb{R})$. Note that in this case $c_{n,s} = n^{\delta+s}$. If $s \in \mathbb{N}$ define ψ_s on $\mathbb{R}$ by letting

$$\psi_s(x) = 0, \quad \text{if } x \in (-\infty, 1), \quad \text{and}$$
$$\psi_s(x) = n^{-\delta/3}, \quad \text{if } x \in [(n-1)^{\delta+s} + 1,\ n^{\delta+s} + 1),\ n \in \mathbb{N}.$$

Then $h * \psi_s \in C_0(\mathbb{R})$, $h * \psi_s = 0$ on $(-\infty, 0]$, and $(h * \psi_s)(x) \geq n^{-\delta/3}$, for $x \in [2, n^{\delta+s})$. If we now use the approach in the preceding case with $h * \psi_s$ in place of g_s we find that

$$\begin{aligned}
\int_{V_{n,s}} (h * \psi_s - \phi_s * h * \psi_s)\,d\lambda &\geq (\zeta(1+\delta/2))^{-1} 2\delta^{-1} n^{-\delta/2} \int_0^{n^{s+\delta}} (h * \psi_s)(t)\,dt \\
&\geq (\zeta(1+\delta/2))^{-1} 2\delta^{-1} n^{-\delta/2} \left\{ \int_0^2 (h * \psi_s)(t)\,dt + \int_2^{n^{s+\delta}} (h * \psi_s)(t)\,dt \right\} \\
&\geq (\zeta(1+\delta/2))^{-1} 2\delta^{-1} n^{-\delta/2} (n^{s+\delta} - 2) n^{-\delta/3} \\
&\geq (\zeta(1+\delta/2))^{-1} \delta^{-1} n^{s+\delta/6}, \quad \text{if } n \geq 4^{1/(s+\delta)}.
\end{aligned}$$

Comparing this with (4.4) with $q = 1$ shows that $h * \psi_s - \phi_s * h * \psi_s \notin \Delta(L^\infty(\mathbb{R}), M(\tau_s, \beta))$. The statements in the theorem for this case now follow by an argument similar to the one for $L^p(\mathbb{R})$, $1 < p < \infty$.

Case II. $p = 1$ and $X = L^1(\mathbb{R})$. If $n, s \in \mathbb{N}$ let $A_{n,s} = (-n^{s+1}, n^{s+1})$. Then $V = A_1^s$, $A_{r+1,s} \cap A_{r,s}^c$ contains a translate of V and $Z(A_{r+1,s}, K_{r+1,s}) \supseteq A_{r,s} + K_{r,s}$. It follows from (4.2) in the proof of Theorem 2 that

$$(4.6) \qquad \sum_{n=1}^{\infty} \left| \int_{A_{n,s}} f \, d\lambda \right| < \infty \quad \text{for all } f \in \Delta(L^1(\mathbb{R}), M(\tau_s)).$$

Now let $\theta_s = (\zeta(2))^{-1} \sum_{i=1}^{\infty} i^{-2} \chi_{(i^{s+1}, i^{s+1}+1)} \in \mathscr{P}^1(\mathbb{R})$, and let $h \in L^1(\mathbb{R})$ be such that $h \geq 0$, $h = 0$ on $(-\infty, 0)$, and $h \neq 0$. We deduce from Lemma 7 and a similar argument to the one used in the previous cases that for all $n \in \mathbb{N}$,

$$\int_{-n^{s+1}}^{n^{s+1}} (h - \theta_s * h)(t) \, dt \geq \zeta(2)^{-1} n^{-1} \int_0^{n^{s+1}} h(t) \, dt.$$

Consequently,

$$\sum_{n=1}^{\infty} \left| \int_{A_{n,s}} (h - \theta_s * h) \, d\lambda \right| = \infty,$$

so that by (4.6) above, $h - \theta_s * h \notin \Delta(L^1(\mathbb{R}), M(\tau_s))$. On the other hand, it is routine to check that $\theta_{s_1} \in M(\tau_{s_2})$ if and only if $s_2 > s_1 + 1$. Hence $h - \theta_{2s} * h \in \Delta(L^1(\mathbb{R}), M(\tau_{2s+2})) \cap (\Delta(L^1(\mathbb{R}), M(\tau_{2s})))^c$, for all $s \in \mathbb{N}$. The conclusion of the theorem for $X = L^1(\mathbb{R})$ now follows along similar lines to the previous cases.

Remark. When G is a noncompact, nondiscrete, σ-compact group which is amenable as a discrete group, it has been proved by E. Granirer [2] (see also related results of W. Rudin in [14] that $L^\infty(G)$ has an invariant mean which is not $\mathscr{P}^1(G)$-invariant. Corresponding results for $C(G)$ have been derived by J. Rosenblatt [11]. The means on these spaces are continuous. Theorem 4 seems to provide the first example of a discontinuous linear functional on a space $L^\infty(G)$ or $C(G)$ which is translation invariant but not $\mathscr{P}^1(G)$-invariant.

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